RIGIDITY OF C^2 INFINITELY RENORMALIZABLE UNIMODAL MAPS

W. DE MELO AND A. A. PINTO

Abstract. Given C^2 infinitely renormalizable unimodal maps f and g with a quadratic critical point and the same bounded combinatorial type, we prove that they are $C^{1+\alpha}$ conjugate along the closure of the corresponding forward orbits of the critical points, for some $\alpha > 0$.

Contents

Key words and phrases. Conjugacy, rigidity, renormalization, unimodal maps.

Stony Brook IMS Preprint #1999/6
May 1999

1. Introduction

It was already clear more than 20 years ago, from the work of Coullet-Tresser and Feigenbaum, that the small scale geometric properties of the orbits of some one dimensional dynamical systems were related to the dynamical behavior of a non-linear operator, the renormalization operator, acting on a space of dynamical systems. This conjectural picture was mathematically established for some classes of analytic maps by Sullivan, McMullen and Lyubich. Here we will extend this description to the space of C^2 maps and prove a rigidity result for a class of unimodal maps of the interval. As it is well-known, a unimodal map is a smooth endomorphism of a compact interval that has a unique critical point which is a turning point. Such a map is renormalizable if there exists an interval neighborhood of the critical point such that the first return map to this interval is again a unimodal map, and the return time is greater than one. The map is infinitely renormalizable if there exist such intervals with arbitrarily high return times. We say that two maps have the same combinatorial type if the map that sends the i-th iterate of the critical point of the first map into the i-th iterate of the critical point of the second map, for all $i \geq 0$, is order preserving. Finally, we say that the combinatorial type of an infinitely renormalizable map is bounded if the ratio of any two consecutive return times is uniformly bounded.

A unimodal map f is C^r with a quadratic critical point if $f = \phi_f \circ p \circ \psi_f$, where $p(x) = x^2$ and ϕ_f, ψ_f are C^r diffeomorphisms. Let c_f be the critical point of f. In this paper we will prove the following rigidity result.

Theorem 1. *Let f and g be C^2 unimodal maps with a quadratic critical point which are infinitely renormalizable and have the same bounded combinatorial type. Then there exists a $C^{1+\alpha}$ diffeomorphism h of the real line such that $h(f^i(c_f)) = g^i(h(c_g))$ for every integer $i \geq 0$.*

We observe that in Theorem 1 the Hölder exponent $\alpha > 0$ depends only upon the bound of the combinatorial type of the maps f and g. Furthermore,as we will see in Section 2, the maps f and g are smoothly conjugated to C^2 normalized unimodal maps $F = \phi_F \circ p$ and $G = \phi_G \circ p$ with critical value 1, and the Hölder constant for the smooth conjugacy between the normalized maps F and G depends only upon the combinatorial type of F and G, and upon the norms $||\phi_F||_{C^2}$ and $||\phi_G||_{C^2}$.

The conclusion of the above rigidity theorem was first obtained by McMullen in [16] under the extra hypothesis that f and g extend to quadratic-like maps in neighborhoods of the dynamical intervals in the complex plane. Combining this last statement with the complex bounds of Levin and van Strien in [11], we get the existence of a $C^{1+\alpha}$ map h which is a conjugacy along the critical orbits for infinitely renormalizable real analytic maps with the same bounded combinatorial type. We extended this result to C^2 unimodal maps in Theorem 1, by combining many results and ideas of Sullivan in [21] with recent results of McMullen in [15], in [16], and of Lyubich in [13] on the hyperbolicity of the renormalization operator R (see the definition of R in the next section). A main lemma used in the proof of Theorem 1 is the following:

Lemma 2. *Let f be a C^2 infinitely renormalizable map with bounded combinatorial type. Then there exist positive constants $\eta < 1$, μ and C, and a real quadratic-like map f_n with conformal modulus greater than or equal to μ, and with the same combinatorial type as the*

n-th renormalization $R^n f$ of f such that

$$||R^n f - f_n||_{C^0} < C\eta^n$$

for every $n \geq 0$.

We observe that in this lemma, the positive constants $\eta < 1$ and μ depend only upon the bound of the combinatorial type of the map f. For normalized unimodal maps f, the positive constant C depends only upon the bound of the combinatorial type of the map f and upon the norm $||\phi_f||_{C^2}$.

This lemma generalizes a Theorem of Sullivan (transcribed as Theorem 4 in Section 2) by adding that the map f_n has the same combinatorial type as the n-th renormalization $R^n f$ of f.

Now, let us describe the proof of Theorem 1 which also shows the relevance of Lemma 2: let f and g be C^2 infinitely renormalizable unimodal maps with the same bounded combinatorial type. Take m to be of the order of a large but fixed fraction of n, and note that $n-m$ is also a fixed fraction of n. By Lemma 2, we obtain a real quadratic-like map f_m exponentially close to $R^m f$, and a real quadratic-like map g_m exponentially close to $R^m g$. Then we use Lemma 6 of Section 2.2 to prove that the renormalization $(n-m)$-th iterates $R^n f$ of $R^m f$, and $R^n g$ of $R^m g$ stay exponentially close to the $(n-m)$-th iterates $R^{n-m} f_m$ of f_m and $R^{n-m} g_m$ of g_m, respectively. Again, by Lemma 2, we have that f_m and g_m have conformal modulus universally bounded away from zero, and have the same bounded combinatorial type of $R^m f$ and $R^m g$. Thus, by the main result of McMullen in [16], the renormalization $(n-m)$-th iterates $R^{n-m} f_m$ of f_m and $R^{n-m} g_m$ of g_m are exponentially close. Therefore, $R^n f$ is exponentially close to $R^{n-m} f_m$, $R^{n-m} f_m$ is exponentially close to $R^{n-m} g_m$, and $R^{n-m} g_m$ is exponentially close to $R^n g$, and so, by the triangle inequality, the n-th iterates $R^n f$ of f and $R^n g$ of g converge exponentially fast to each other. Finally, by Theorem 9.4 in the book [18] of de Melo and van Strien, we conclude that f and g are $C^{1+\alpha}$ conjugate along the closure of their critical orbits.

Let us point out the main ideas in the proof of Lemma 2: Sullivan in [21] proves that $R^n f$ is exponentially close to a quadratic-like map F_n which has conformal modulus universally bounded away from zero. The quadratic-like map F_n determines a unique quadratic map $P_{c(F_n)}(z) = 1 - c(F_n)z^2$ which is hybrid conjugated to F_n by a K quasiconformal homeomorphism, where K depends only upon the conformal modulus of F_n (see Theorem 1 of Douady-Hubbard in [6], and Lemma 11 in Section 3.3). In [13], Lyubich proves the bounded geometry of the Cantor set consisting of all the parameters of the quadratic family $P_c(z) = 1 - cz^2$ corresponding to infinitely renormalizable maps with combinatorial type bounded by N (see definition in Section 2 and the proof of Lemma 2). In Lemma 8 of Section 2.2, we prove that $R^n f$ and F_n have exponentially close renormalization types. Therefore, letting c_n be the parameter corresponding to the quadratic map P_{c_n} with the same combinatorial type as $R^n f$, we have, from the above result of Lyubich, that $c(F_n)$ and c_n are exponentially close. In Lemma 12 of Section 3.3, we use holomorphic motions to prove the existence of a real quadratic-like map f_n which is hybrid conjugated to P_{c_n}, and has the following essential property: the distance between F_n and f_n is proportional to the distance between $c(F_n)$ and c_n raised to some positive constant. Therefore, the real quadratic-like map f_n has the same combinatorial type as $R^n f$, and f_n is exponentially close to F_n. Since the map F_n is exponentially close to $R^n f$, we obtain that the map f_n is also exponentially close to $R^n f$.

The example of Faria and de Melo in [7] for critical circle maps can be adapted to prove the existence of a pair of C^∞ unimodal maps, with the same unbounded combinatorial type, such that the conjugacy h has no $C^{1+\alpha}$ extension to the reals for any $\alpha > 0$.

2. Shadowing unimodal maps

A C^r unimodal map $F : I \to I$ is *normalized* if $I = [-1, 1]$, $F = \phi_F \circ p$, $F(0) = 1$, and $\phi_F : [0, 1] \to I$ is a C^r diffeomorphism. A C^r unimodal map $f = \phi_f \circ p \circ \psi_f$ with quadratic critical point either has trivial dynamics or has an invariant interval where it is C^r conjugated to a C^r normalized unimodal map F. Take, for instance, the map

$$\phi_F(x) = \left(\psi_f^{-1} \circ \phi_f(0)\right)^{-1} \cdot \psi_f^{-1} \circ \phi_f \left(\left(\psi_f^{-1} \circ \phi_f(0)\right)^{-2} \cdot x\right) .$$

Therefore, from now on we will only consider C^r normalized unimodal maps f.

The map f is *renormalizable* if there is a closed interval J centered at the origin, strictly contained in I, and $l > 1$ such that the intervals $J, \ldots, f^{l-1}(J)$ are disjoint, $f^l(J)$ is strictly contained in J and $f^l(0) \in \partial J$. If f is renormalizable, we always consider the smallest $l > 1$ and the minimal interval $J_f = J$ with the above properties. The set of all renormalizable maps is an open set in the C^0 topology. The *renormalization operator* R acts on renormalizable maps f by $Rf = \psi \circ f^l \circ \psi^{-1} : I \to I$, where $\psi : J_f \to I$ is the restriction of a linear map sending $f^l(0)$ into 1. Inductively, the map f is n *times renormalizable* if $R^{n-1}f$ is renormalizable. If f is n times renormalizable for every $n > 0$, then f is *infinitely renormalizable*.

Let f be a renormalizable map. We label the intervals $J_f, \ldots, f^{l-1}(J_f)$ of f by $1, \ldots, l$ according to their embedding on the real line, from the left to the right. *The permutation* $\sigma_f : \{1, \ldots, l\} \to \{1, \ldots, l\}$ is defined by $\sigma_f(i) = j$ if the interval labeled by i is mapped by f to the interval labeled by j. The *renormalization type* of an n times renormalizable map f is given by the sequence $\sigma_f, \ldots, \sigma_{R^n f}$. An n times renormalizable map f has *renormalization type bounded by* $N > 1$ if the number of elements of the domain of each permutation $\sigma_{R^m f}$ is less than or equal to N for every $0 \le m \le n$. We have the analogous notions for infinitely renormalizable maps.

Note that if any two maps are n times renormalizable and have the same combinatorial type (see definition in the introduction), then they have the same renormalization type. The converse is also true in the case of infinitely renormalizable maps. An infinitely renormalizable map has combinatorial type bounded by $N > 1$ if the renormalization type is bounded by N.

If $f = \phi \circ p$ is n times renormalizable, and $\phi \in C^2$, there is a C^2 diffeomorphism ϕ_n satisfying $R^n f = \phi_n \circ p$. The *nonlinearity* $\mathrm{nl}(\phi_n)$ of ϕ_n is defined by

$$\mathrm{nl}(\phi_n) = \sup_{x \in p(I)} \left| \frac{\phi_n''(x)}{\phi_n'(x)} \right| .$$

Let $\mathcal{I}(N, b)$ be the set of all C^2 normalized unimodal maps $f = \phi \circ p$ with the following properties:

(i) f is infinitely renormalizable;
(ii) the combinatorial type of f is bounded by N;
(iii) $||\phi||_{C^2} \le b$.

Theorem 3. (Sullivan [21]) *There exist positive constants B and $n_1(b)$ such that, for every $f \in \mathcal{I}(N, b)$, the n-th renormalization $R^n f = \phi_n \circ p$ of f has the property that $\mathrm{nl}(\phi_n) \leq B$ for every $n \geq n_1$.*

This theorem together with Arzelá-Ascoli's Theorem implies that, for every $0 \leq \beta < 2$, and for every $n \geq n_1(b)$, the renormalization iterates $R^n f$ are contained in a compact set of unimodal maps with respect to the C^β norm. We will use this fact in the proof of Lemma 5 below.

2.1. **Quadratic-like maps.** A *quadratic-like map* $f : V \to W$ is a holomorphic map with the property that V and W are simply connected domains with the closure of V contained in W, and f is a degree two branched covering map. We add an extra condition that f has a continuous extension to the boundary of V. The *conformal modulus of a quadratic-like map* $f : V \to W$ is equal to the conformal modulus of the annulus $W \setminus \overline{V}$. A *real quadratic-like map* is a quadratic-like map which commutes with complex conjugation.

The *filled Julia set* $\mathcal{K}(f)$ *of* f is the set $\{z : f^n(z) \in V, \text{ for all } n \geq 0\}$. Its boundary is the *Julia set* $\mathcal{J}(f)$ of f. These sets $\mathcal{J}(f)$ and $\mathcal{K}(f)$ are connected if the critical point of f is contained in $\mathcal{K}(f)$.

Let $\mathcal{Q}(\mu)$ be the set of all real quadratic-like maps $f : V \to W$ satisfying the following properties:

(i) the Julia set $\mathcal{J}(f)$ of f is connected;
(ii) the conformal modulus of f is greater than or equal to μ, and less than or equal to 2μ;
(iii) f is normalized to have the critical point at the origin, and the critical value at one.

By Theorem 5.8 in page 72 of [15], the set $\mathcal{Q}(\mu)$ is compact in the Carathéodory topology taking the critical point as the base point (see definition in page 67 of [15]).

Theorem 4. (Sullivan [21]) *There exist positive constants $\gamma(N) < 1$, $C(b, N)$, and $\mu(N)$ with the following property: if $f \in \mathcal{I}(N, b)$, then there exists $f_n \in \mathcal{Q}(\mu)$ such that $||R^n f - f_n||_{C^0} \leq C\gamma^n$.*

In the following sections, we will develop the results that will be used in the last section to prove the generalization of Theorem 4 (as stated in Lemma 2), and to prove Theorem 1.

2.2. **Maps with close combinatorics.** Let $D(\sigma)$ be the open set of all C^0 renormalizable unimodal maps f with renormalization type $\sigma_f = \sigma$. The open sets $D(\sigma)$ are pairwise disjoint. Let $E(\sigma)$ be the complement of $D(\sigma)$ in the set of all C^0 unimodal maps f.

Lemma 5. *There exist positive constants $n_2(b)$ and $\epsilon(N)$ with the following property: for every $f \in \mathcal{I}(N, b)$, for every $n \geq n_2$, and for every $g \in E(\sigma_{R^n f})$, we have $||R^n f - g||_{C^0} > \epsilon$.*

Proof. Suppose, by contradiction, that there is a sequence $R^{m_1} f_1$, $R^{m_2} f_2$, … with the property that for a chosen σ there is a sequence $g_1, g_2, \ldots \in E(\sigma)$ satisfying $||R^{m_i} f_i - g_i||_{C^0} < 1/i$. By Theorem 3, there are $B > 0$ and $n_1(b) \geq 1$ such that the maps $R^{m_i} f_i$ have nonlinearity bounded by $B > 0$ for all $m_i \geq n_1$. By Arzela-Ascoli's Theorem, there is a subsequence $R^{m_{i_1}} f_{i_1}, R^{m_{i_2}} f_{i_2}, \ldots$ which converges in the C^0 topology to a map g. Hence, the map g is contained in the boundary of $E(\sigma)$ and is infinitely renormalizable. However, a map contained in the boundary of $E(\sigma)$ is not renormalizable, and so we get a contradiction. $\square$

Lemma 6. *There exist positive constants $n_3(N,b)$ and $L(N)$ with the following property: for every $f \in \mathcal{I}(N,b)$, for every C^2 renormalizable unimodal map g, and for every $n > n_3$, we have*

$$||R^n f - Rg||_{C^0} \leq L||R^{n-1}f - g||_{C^0} \ .$$

Proof. In the proof of this lemma we will use the inequality (1) below. Let $f_1, \ldots, f_m$ be maps with C^1 norm bounded by some constant $d > 0$, and let $g_1, \ldots, g_m$ be C^0 maps. By induction on m, and by the Mean Value Theorem, there is $c(m,d) > 0$ such that

$$||f_1 \circ \ldots \circ f_m - g_1 \circ \ldots \circ g_m||_{C^0} \leq c \max_{i=1,\ldots,m} \{||f_i - g_i||_{C^0}\} \ . \tag{1}$$

Set $n_3 = \max\{n_1, n_2\}$, where $n_1(b)$ is defined as in Theorem 3, and $n_2(b)$ is defined as in Lemma 5. Set $F = R^{n-1}f$ with $n \geq n_3$. We start by considering the simple case (a), where F and g do not have the same renormalization type, and conclude with the complementary case (b). In case (a), by Lemma 5, there is $\epsilon(N) > 0$ with the property that

$$||RF - Rg||_{C^0} \leq 2 \leq 2\epsilon^{-1}||F - g||_{C^0} \ .$$

In case (b), there is $1 < m \leq N$ such that $RF(x) = a_F F^m(a_F^{-1}x)$, and $Rg(x) = a_g g^m(a_g^{-1}x)$, where $a_F = F^m(0)$ and $a_g = g^m(0)$. By Theorem 3, there is a positive constant $B(N)$ bounding the nonlinearity of F. Since the set of all infinitely renormalizable unimodal maps F with nonlinearity bounded by B is a compact set with respect to the C^0 topology, and since a_F varies continuously with F, there is $S(N) > 0$ with the property that $|a_F| \geq S$. Again, by Theorem 3, and by inequality (1), there is $c_1(N) > 0$ such that

$$||F^m - g^m||_{C^0} \leq c_1||F - g||_{C^0} \ . \tag{2}$$

Thus,

$$|a_F - a_g| \leq c_1||F - g||_{C^0} \ . \tag{3}$$

Now, let us consider the cases where (i) $||F - g|| \geq S/(2c_1)$ and (ii) $||F - g|| \leq S/(2c_1)$. In case (i), we get

$$||RF - Rg||_{C^0} \leq 2 \leq 4c_1 S^{-1}||F - g||_{C^0} \ .$$

In case (ii), using that $|a_F| \geq S$ and (3), we get $a_g \geq a_F - S/2 \geq S/2$, and thus, by (2), we obtain

$$\left|a_F^{-1} - a_g^{-1}\right| \leq a_F^{-1}a_g^{-1}|a_F - a_g| \leq 2S^{-2}c_1||F - g||_{C^0} \ .$$

Hence, again by (2) and (3), there is $c_2(N) > 0$ with the property that

$$\begin{aligned} ||RF - Rg||_{C^0} &\leq ||F^m||_{C^0}|a_F - a_g| + |a_g|||F^m||_{C^1}\left|a_F^{-1} - a_g^{-1}\right| \\ &\quad + |a_g|||F^m - g^m||_{C^0} \\ &\leq c_2||F - g||_{C^0} \ . \end{aligned}$$

Therefore, this lemma is satisfied with $L(N) = \max\{2\epsilon^{-1}, 4c_1S^{-1}, c_2\}$. □

Lemma 7. *For all positive constants $\lambda < 1$ and C there exist positive constants $\alpha(N,\lambda)$ and $n_4(b,N,\lambda,C)$ with the following property: for every $f \in \mathcal{I}(N,b)$, and every $n > n_4$, if f_n is a C^2 unimodal map such that*

$$||R^n f - f_n||_{C^0} < C\lambda^n \ ,$$

then f_n is $[\alpha n + 1]$ times renormalizable with $\sigma_{R^m f_n} = \sigma_{R^{n+m}f}$ for every $m = 0, \ldots, [\alpha n]$ (where $[y]$ means the integer part of $y > 0$.)

Proof. Let $\epsilon(N)$ and $n_2(b)$ be as defined in Lemma 5, and let $L(N)$ and $n_3(b)$ be as defined in Lemma 6. Take $\alpha > 0$ such that $L^\alpha \lambda < 1$. Set $n_4 \geq \max\{n_2, n_3\}$ such that $C\lambda^{n_4} < \epsilon$ and $C\lambda^{n_4} L^{[\alpha n_4]} < \epsilon$. Then, for every $n > n_4$, the values $C\lambda^n, C\lambda^n L, \ldots, C\lambda^n L^{[\alpha n]}$ are less than ϵ.

By Lemma 5, if $||R^n f - f_n||_{C^0} < C\lambda^n < \epsilon$ with $n > n_4$, then the map f_n is contained in $D(\sigma_{R^n f})$. Thus, f_n is once renormalizable, and $\sigma_{f_n} = \sigma_{R^n f}$. By induction on $m = 1, \ldots, [\alpha n]$, let us suppose that f_n is m times renormalizable, and $\sigma_{R^i f_n} = \sigma_{R^{n+i} f}$ for every $i = 0, \ldots, m-1$. By Lemma 6, we get that $||R^{n+m} f - R^m f_n||_{C^0} < CL^m \lambda^n < \epsilon$. Hence, again by Lemma 5, the map $R^m f_n$ is once renormalizable, and $\sigma_{R^m f_n} = \sigma_{R^{n+m} f}$. □

Lemma 8. *There exist positive constants $\gamma(N) < 1$, $\alpha(N)$, $\mu(N)$, and $C(b, N)$ with the following property: for every $f \in \mathcal{I}(N, b)$, there exists $f_n \in \mathcal{Q}(\mu)$ such that*

(i) $||R^n f - f_n||_{C^0} \leq C\gamma^n$;

(ii) f_n *is* $[\alpha n + 1]$ *times renormalizable with* $\sigma_{R^m f_n} = \sigma_{R^{n+m} f}$ *for every* $m = 0, \ldots, [\alpha n]$.

Proof. The proof follows from Theorem 4 and Lemma 7. □

3. Varying quadratic-like maps

We start by introducing some classical results on Beltrami differentials and holomorphic motions, all of which we will apply later in this section to vary the combinatorics of quadratic-like maps.

3.1. **Beltrami differentials.** A homeomorphism $h : U \to V$, where U and V are contained in $\mathbb{C}$ or $\overline{\mathbb{C}}$, is *quasiconformal* if it has locally integrable distributional derivatives ∂h, $\overline{\partial} h$, and if there is $\epsilon < 1$ with the property that $\left|\overline{\partial} h / \partial h\right| \leq \epsilon$ almost everywhere. The Beltrami differential μ_h of h is given by $\mu_h = \overline{\partial} h / \partial h$. A quasiconformal map h is K *quasiconformal* if $K \geq (1 + ||\mu_h||_\infty)/(1 - ||\mu_h||_\infty)$.

We denote by $D_R(c_0)$ the open disk in $\mathbb{C}$ centered at the point c_0 and with radius $R > 0$. We also use the notation $D_R = D_R(0)$ for the disk centered at the origin.

The following theorem is a slight extension of Theorem 4.3 in page 27 of the book [9] by Lehto.

Theorem 9. *Let $\psi : \mathbb{C} \to \mathbb{C}$ be a quasiconformal map with the following properties:*

(i) $\mu_\psi = \overline{\partial}\psi / \partial\psi$ *has support contained in the disk* D_R;

(ii) $||\mu_\psi||_\infty < \epsilon < 1$;

(iii) $\lim_{|z| \to \infty} (\psi(z) - z) = 0$.

Then there exists $C(\epsilon, R) > 0$ such that

$$||\psi - id||_{C^0} \leq C ||\mu_\psi||_\infty \ .$$

Proof. Let us define $\phi_1 = \mu_\psi$, and, by induction on $i \geq 1$, we define $\phi_{i+1} = \mu_\psi H \phi_i$, where $H\phi_i$ is the Hilbert transform of ϕ_i given by the Cauchy Principal Value of

$$\frac{-1}{\pi} \int\int_{\mathbb{C}} \frac{\phi_i(\xi)}{(\xi - z)^2} du dv \ .$$

By Theorem 4.3 in page 27 of [9], we get $\psi(z) = z + \sum_{i=1}^{\infty} T\phi_i(z)$, where $T\phi_i(z)$ is given by

$$\frac{-1}{\pi} \int\int_{\mathbb{C}} \frac{\phi_i(\xi)}{\xi - z} du dv \ .$$

By the Calderón-Zigmund inequality (see page 27 of [9]), for every $p \geq 1$, the Hilbert operator $H : L^p \to L^p$ is bounded, and its norm $||H||_p$ varies continuously with p. An elementary integration also shows that $||H||_2 = 1$ (see page 157 of [10]). Therefore, given that $||\mu_\psi||_\infty < \epsilon$, there is $p_0(\epsilon) > 2$ with the property that

$$||H||_{p_0} ||\mu_\psi||_\infty < ||H||_{p_0} \epsilon < 1 \ . \tag{4}$$

Since $p_0 > 2$, it follows from Hölder's inequality (see page 141 of [10]) that there is a positive constant $c_1(p_0, R)$ such that

$$||T\phi_i||_{C^0} \leq c_1 ||\phi_i||_{p_0} \ . \tag{5}$$

By a simple computation, we get

$$||\phi_i||_{p_0} \leq (\pi R^2)^{\frac{1}{p_0}} ||H||_{p_0}^{i-1} ||\mu_\psi||_\infty^i \ . \tag{6}$$

Thus, by inequalities (4), (5), and (6), there is a positive constant $c_2(\epsilon, R)$ with the property that

$$\begin{aligned} ||\psi - id||_{C^0} &\leq \sum_{i=1}^{\infty} ||T\phi_i||_{C^0} \leq \frac{c_1 (\pi R^2)^{\frac{1}{p_0}} ||\mu_\psi||_\infty}{1 - ||H||_{p_0} ||\mu_\psi||_\infty} \\ &\leq c_2 ||\mu_\psi||_\infty \ . \end{aligned}$$

□

3.2. **Holomorphic motions.** A *holomorphic motion of a subset X of the Riemann sphere over a disk $D_R(c_0)$* is a family of maps $\psi_c : X \to X_c$ with the following properties: (i) ψ_c is an injection of X onto a subset X_c of the Riemann sphere; (ii) $\psi_{c_0} = id$; (iii) for every $z \in X$, $\psi_c(z)$ varies holomorphically with $c \in D_R(c_0)$.

Theorem 10. **(Słodkowski** [23]**)** *Let $\psi_c : X \to X_c$ be a holomorphic motion over the disk $D_R(c_0)$. Then there is a holomorphic motion $\Psi_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ over the disk $D_R(c_0)$ such that*

(i) $\Psi_c | X = \psi_c$;
(ii) Ψ_c is a K_c quasiconformal map with

$$K_c = \frac{R + |c - c_0|}{R - |c - c_0|} \ .$$

See also Douady's survey [5].

3.3. **Varying the combinatorics.** Let $\mathcal{M}$ be the set of all quadratic-like maps with connected Julia set. Let $\mathcal{P}$ be the set of all normalized quadratic maps $P_c : \mathbb{C} \to \mathbb{C}$ defined by $P_c(z) = 1 - cz^2$, where $c \in \mathbb{C} \setminus \{0\}$. Two quadratic-like maps f and g are *hybrid conjugate* if there is a quasiconformal conjugacy h between f and g with the property that $\overline{\partial} h(z) = 0$ for almost every $z \in \mathcal{K}(f)$. By Douady-Hubbard's Theorem 1 in page 296 of [6], for every $f \in \mathcal{M}$ there exists a unique quadratic map $P_{c(f)}$ which is hybrid conjugated to f. The map $\xi : \mathcal{M} \to \mathcal{P}$ defined by $\xi(f) = P_{c(f)}$ is called the *straightening*.

Observe that a real quadratic map P_c with $c \notin [1, 2]$ has trivial dynamics. Therefore, we will restrict our study to the set $\mathcal{Q}([1, 2], \mu)$ of all $f \in \mathcal{Q}(\mu)$ satisfying $\xi(f) = P_{c(f)}$ for some $c(f) \in [1, 2]$.

Let us choose a radius Δ large enough such that, for every $c \in [1, 2]$, $P_c(z) = 1 - cz^2$ is a quadratic-like map when restricted to $P_c^{-1}(D_\Delta)$.

Lemma 11. *There exist positive constants $\Omega(\mu)$ and $K(\mu)$ with the following property: for every $f \in \mathcal{Q}([1,2],\mu)$ there exists a topological disk $V_f \subset D_\Omega$ such that f restricted to $f^{-1}(V_f)$ is a quadratic-like map. Furthermore, there is a K quasiconformal homeomorphism $\Phi_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that*

(i) $\Phi_f|\Phi_f^{-1}(V_f)$ is a hybrid conjugacy between f and $P_{c(f)}$;
(ii) $\Phi_f(V_f) = D_\Delta$;
(iii) Φ_f is holomorphic over $\overline{\mathbb{C}} \setminus \overline{V_f}$;
(iv) $\Phi_f(\overline{z}) = \overline{\Phi_f(z)}$.

Proof. The main point in this proof is to combine the hybrid conjugacy between f and $P_{c(f)}$ given by Douady-Hubbard, with Sullivan's pull-back argument, and with McMullen's rigidity theorem for real quadratic maps. Using Sullivan's pull-back argument and the hybrid conjugacy between f and $P_{c(f)}$, we construct a K quasiconformal homeomorphism $\Phi_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which restricts to a conjugacy between f and $P_{c(f)}$. Moreover, Φ_f satisfies properties (ii), (iii) and (iv) of this lemma, and the restriction of Φ_f to the filled in Julia set of f extends to a quasi conformal map that is a hybrid conjugacy between f and $P_{c(f)}$. By Rickman's glueing lemma (see Lemma 2 in [6]) it follows that Φ_f also satisfies property (i) of this lemma.

Now, we give the details of the proof: let us consider the set of all quadratic-like maps $f : W_f \to W'_f$ contained in $\mathcal{Q}([1,2],\mu)$. Using the Koebe Distortion Lemma (see page 84 of [2]), we can slightly shrink $f^{-n}(W'_f)$ for some $n \geq 0$ to obtain an open set V_f with the following properties:

(i) V_f is symmetric with respect to the real axis;
(ii) the restriction of f to $f^{-1}(V_f)$ is a quadratic-like map;
(iii) the annulus $V_f \setminus \overline{f^{-1}(V_f)}$ has conformal modulus between $\mu/2$ and 2μ;
(iv) the boundaries of $V_f \setminus \overline{f^{-1}(V_f)}$ are analytic $\gamma(\mu)$ quasi-circles for some $\gamma(\mu) > 0$, i. e., they are images of an Euclidean circle by $\gamma(\mu)$ quasiconformal maps defined on $\overline{\mathbb{C}}$.

Let $\mathcal{Q}'$ be the set of all quadratic-like maps $f : f^{-1}(V_f) \to V_f$ contained in $\mathcal{Q}([1,2],\mu/2) \cup \mathcal{Q}([1,2],\mu)$ for which V_f satisfies properties (i), ..., (iv) of last paragraph. Since for every $f \in \mathcal{Q}'$ the boundaries of $V_f \setminus \overline{f^{-1}(V_f)}$ are analytic $\gamma(\mu)$ quasi-circles, any convergent sequence $f_n \in \mathcal{Q}'$, with limit g, in the Carathéodory topology has the property that the sets V_{f_n} converge to V_g in the Hausdorff topology (see Section 4.1 in pages 75-76 of [16]). Therefore, the set $\mathcal{Q}'$ is closed with respect to the Carathéodory topology, and hence is compact. Furthermore, by compactness of $\mathcal{Q}'$, and using the Koebe Distortion Lemma, there is an Euclidean disk D_Ω which contains V_f for every $f \in \mathcal{Q}'$.

Now, let us construct $\Phi_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that the properties (i), ..., (iv) of this lemma are satisfied.

Since V_f is symmetric with respect to the real axis, there is a unique Riemann Mapping $\phi : \overline{\mathbb{C}} \setminus \overline{V_f} \to \overline{\mathbb{C}} \setminus \overline{D_\Delta}$ satisfying $\phi(\overline{z}) = \overline{\phi(z)}$, and such that $\phi(\mathbb{R}^+) \subset \mathbb{R}^+$. Since the boundaries of $V_f \setminus \overline{f^{-1}(V_f)}$ are analytic $\gamma(\mu)$ quasi-circles, using the Ahlfors-Beurling Theorem (see Theorem 5.2 in page 33 of [9]) the map ϕ has a $K_1(\mu)$ quasiconformal homeomorphic extension $\phi_1 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ which also is symmetric $\phi_1(\overline{z}) = \overline{\phi_1(z)}$.

Let $\phi_2 : V_f \setminus \mathcal{K}(f) \to D_\Delta \setminus \mathcal{K}(P_{c(f)})$ be the unique continuous lift of ϕ_1 satisfying $P_{c(f)} \circ \phi_2(z) = \phi_1 \circ f(z)$, and such that $\phi_2(\mathbb{R}^+) \subset \mathbb{R}^+$. Since ϕ_1 is a $K_1(\mu)$ quasiconformal homeomorphism, so is ϕ_2.

Using the Ahlfors-Beurling Theorem, we construct a $K_2(\mu)$ quasi-conformal homeomorphism $\phi_3 : \overline{\mathbb{C}} \setminus \mathcal{K}(f) \to \overline{\mathbb{C}} \setminus \mathcal{K}(P_{c(f)})$ interpolating ϕ_1 and ϕ_2 with the following properties:

(i) $\phi_3(z) = \phi_1(z)$ for every $z \in \overline{\mathbb{C}} \setminus V_f$;
(ii) $\phi_3(z) = \phi_2(z)$ for every $z \in \overline{f^{-1}(V_f)} \setminus \mathcal{K}(f)$;
(iii) $\phi_3(\overline{z}) = \overline{\phi_3(z)}$.

Then the map ϕ_3 conjugates f on $\partial f^{-1}(V_f)$ with $P_{c(f)}$ on $\partial P^{-1}_{c(f)}(D_\Delta)$, and is holomorphic over $\overline{\mathbb{C}} \setminus \overline{V_f} \subset \overline{\mathbb{C}} \setminus \overline{D_\Omega}$.

By Theorem 1 in [6], there is a K'_f quasiconformal hybrid conjugacy $\phi_4 : V'_f \to V'_{c(f)}$ between f and $P_{c(f)}$, where V'_f is a neigbourhood of $\mathcal{K}(f)$. Using the Ahlfors-Beurling Theorem, we construct a K''_f quasiconformal homeomorphism $\Phi_0 : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ interpolating ϕ_3 and ϕ_4 such that

(i) $\Phi_0(z) = \phi_3(z)$ for every $z \in \overline{\mathbb{C}} \setminus f^{-1}(V_f)$;
(ii) $\Phi_0(z) = \phi_4(z)$ for every $z \in \mathcal{K}(f)$;
(iii) $\Phi_0(\overline{z}) = \overline{\Phi_0(z)}$.

Then the map Φ_0 conjugates f on $\mathcal{K}(f) \cup \partial f^{-1}(V_f)$ with $P_{c(f)}$ on $\mathcal{K}(P_{c(f)}) \cup \partial P^{-1}_{c(f)}(D_\Delta)$, and satisfies the properties (ii), (iii) and (iv) as stated in this lemma. Furthermore, $\mu_{\Phi_0}(z) = 0$ for every $z \in \overline{\mathbb{C}} \setminus V_f$, $|\mu_{\Phi_f}(z)| \leq (K_2 - 1)/(K_2 + 1)$ for a. e. $z \in V_f \setminus f^{-1}(V_f)$, and $\mu_{\Phi_f}(z) = 0$ for a. e. $z \in \mathcal{K}(f) \setminus \mathcal{J}(f)$.

For every $n > 0$, let us inductively define the K''_f quasiconformal homeomorphism $\Phi_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ as follows:

(i) $\Phi_n(z) = \Phi_{n-1}(z)$ for every $z \in \left(\overline{\mathbb{C}} \setminus f^{-n}(V_f)\right) \cup \mathcal{K}(f)$;
(ii) $P_{c(f)} \circ \Phi_n(z) = \Phi_{n-1} \circ f(z)$ for every $z \in f^{-n}(V_f) \setminus \mathcal{K}(f)$.

By compactness of the set of all K''_f quasiconformal homeomorphisms on $\overline{\mathbb{C}}$ fixing three points (0, 1 and ∞), there is a subsequence Φ_{n_j} which converges to a K''_f quasiconformal homeomorphism Φ_f. Then Φ_f satisfies the properties (ii), (iii) and (iv) as stated in this lemma.

The restriction of Φ_f to the set $f^{-1}(V_f)$ has the property of being a quasiconformal conjugacy between f and $P_{c(f)}$. Furthermore, the Beltrami differential μ_{Φ_f} has the following properties:

(i) $\mu_{\Phi_f}(z) = 0$ for every $z \in \overline{\mathbb{C}} \setminus V_f$;
(ii) $|\mu_{\Phi_f}(z)| \leq (K_2 - 1)/(K_2 + 1)$ for a. e. $z \in V_f \setminus \mathcal{K}(f)$;
(iii) $\mu_{\Phi_f}(z) = 0$ for a. e. $z \in \mathcal{K}(f) \setminus \mathcal{J}(f)$.

Therefore, by Rickman's glueing lemma, $\Phi_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a $K_2(\mu)$ quasiconformal homeomorphism, and Φ_f restricted to the set $f^{-1}(V_f)$ is a hybrid conjugacy between f and $P_{c(f)}$. □

The lemma below could be proven using the external fibers and the fact that the holonomy of the hybrid foliation is quasi conformal as in [13]. However we will give a more direct proof of it below.

Lemma 12. *There exist positive constants $\beta(\mu) \leq 1$, $D(\mu)$, and $\mu'(\mu)$ with the following property: for every $c \in [1, 2]$, and for every $f \in \mathcal{Q}([1, 2], \mu)$, there is $f_c \in \mathcal{Q}([1, 2], \mu')$*

satisfying $\xi(f_c) = P_c$, *and such that*

$$||f - f_c||_{C^0(I)} \leq D|c(f) - c|^\beta \ . \tag{7}$$

Proof. The main step of this proof consists of constructing the real quadratic-like maps $f_c = \psi_c \circ P_c \circ \psi_c^{-1}$ satisfying $f_{c(f)} = f$, and such that the maps $\omega_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by $\omega_c = \psi_c \circ \psi_{c(f)}^{-1}$ form a holomorphic motion ω_c, and have the property of being holomorphic on the complement of a disk centered at the origin. Using Theorem 9 and Theorem 10, we prove that there is a positive constant L_3 with the property that $||\omega_c - id||_{C^0} \leq L_3|c - c(f)|$. Finally, we show that this implies the inequality (7) above.

Now, we give the details of the proof: let us choose a small $\epsilon > 0$, and a small open set U of $\mathbb{C}$ containing the interval $[1, 2]$ such that, for every $c \in U$, the quadratic map $P_c(z) = 1 - cz^2$ has a quadratic-like restriction to $P_c^{-1}(D_\Delta)$, and $P_c^{-1}(D_\Delta) \subset D_{\Delta-\epsilon}$. Let $\eta : \mathbb{C} \to \mathbb{R}$ be a C^∞ function with the following properties:

(i) $\eta(z) = 1$ for every $z \in \mathbb{C} \setminus D_\Delta$;
(ii) $\eta(z) = 0$ for every $z \in D_{\Delta-\epsilon}$;
(iii) $\eta(z) = \eta(\overline{z})$ for every $z \in \mathbb{C}$.

There is a unique continuous lift $\alpha_c : \mathbb{C} \setminus P_{c_0}^{-1}(D_\Delta) \to \mathbb{C} \setminus P_c^{-1}(D_\Delta)$ of the identity map such that

(i) $P_c \circ \alpha_c(z) = P_{c_0}(z)$;
(ii) $\alpha_{c_0} = id$;
(iii) $\alpha_c(z)$ varies continuously with c.

Then the maps α_c are holomorphic injections, and, for every $z \in \mathbb{C} \setminus P_{c_0}^{-1}(D_\Delta)$, $\alpha_c(z)$ varies holomorphically with c.

Let $\beta_c : \mathbb{C} \setminus P_{c_0}^{-1}(D_\Delta) \to \mathbb{C} \setminus P_c^{-1}(D_\Delta)$ be the interpolation between the identity map and α_c defined by $\beta_c = \eta \cdot id + (1 - \eta) \cdot \alpha_c$. We choose $r' > 0$ small enough such that, for every $c_0 \in [1, 2]$, and $c \in D_{r'}(c_0) \subset U$, β_c is a diffeomorphism. Then $\beta_c : \mathbb{C} \setminus P_{c_0}^{-1}(D_\Delta) \to \mathbb{C} \setminus P_c^{-1}(D_\Delta)$ is a holomorphic motion over $D_r(c_0)$ with the following properties:

(i) the map β_c is a conjugacy between P_{c_0} on $\partial P_{c_0}^{-1}(D_\Delta)$ and P_c on $\partial P_c^{-1}(D_\Delta)$;
(ii) the restriction of β_c to the set $\mathbb{C} \setminus D_\Delta$ is the identity map;
(iii) if c is real then $\beta_c(\overline{z}) = \overline{\beta_c(z)}$.

By Theorem 10, β_c extends to a holomorphic motion $\hat{\beta}_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ over $D_{r'}(c_0)$, and, by taking $r = r'/2$, the map $\hat{\beta}_c$ is 3 quasiconformal for every $c \in D_r(c_0)$.

By Lemma 11, there is a $K(\mu)$ quasiconformal homeomorphism $\Phi_f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, and an open set $V_f = \Phi_f^{-1}(D_\Delta)$ such that (i) Φ_f restricted to $f^{-1}(V_f)$ is a hybrid conjugacy between f and $P_{c(f)}$; (ii) Φ_f is holomorphic over $\overline{\mathbb{C}} \setminus \overline{V_f}$; and (iii) $\Phi_f(\overline{z}) = \overline{\Phi_f(z)}$. Let $\Phi_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be defined by $\Phi_c = \hat{\beta}_c \circ \Phi_f$. Then, for every $c \in D_r(c_0)$, Φ_c is a $3K$ quasiconformal homeomorphism which conjugates f on $\partial f^{-1}(V_f)$ with P_c on $\partial P_c^{-1}(D_\Delta)$.

We define the Beltrami differential μ_c as follows:

(i) $\mu_c(z) = 0$ if $z \in \mathcal{K}(P_c) \cup (\mathbb{C} \setminus D_\Delta)$;
(ii) $(\Phi_c)^* \mu_c(z) = 0$ if $z \in D_\Delta \setminus \overline{P_c^{-1}(D_\Delta)}$;
(iii) $(P_c^n)_* \mu_c(z) = \mu_c(P_c^n(z))$ if $z \in P_c^{-n}(D_\Delta) \setminus \overline{P_c^{-(n+1)}(D_\Delta)}$ and $n \geq 1$.

Then (i) the Beltrami differential μ_c varies holomorphically with c; (ii) $||\mu_c||_\infty < (3K-1)/(3K+1)$ for every $c \in D_r(c(f))$; and (iii) if c is real then $\mu_c(\overline{z}) = \overline{\mu_c(z)}$ for almost every $z \in \mathbb{C}$.

By the Ahlfors-Bers Theorem (see [3]), for every $c \in D_r(c(f))$ there is a normalized $3K$ quasiconformal homeomorphism $\psi_c : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $\psi_c(0) = 0$, $\psi_c(1) = 1$, and $\psi_c(\infty) = \infty$ such that $\mu_{\psi_c} = \mu_c$, and $\psi_c(z)$ varies holomorphically with c. Thus, the restriction of ψ_c to $\overline{\mathbb{C}} \setminus \overline{D_\Delta}$ is a holomorphic map, and if c is real then $\psi_c(\overline{z}) = \overline{\psi_c(z)}$ for every $z \in \mathbb{C}$.

The map $f_c : \psi_c(P_c^{-1}(D_\Delta)) \to \psi_c(D_\Delta)$ defined by $f_c = \psi_c \circ P_c \circ \psi_c^{-1}$ is 1 quasiconformal, and thus a holomorphic map. Furthermore, the map f_c is hybrid conjugated to P_c, and so f_c is a quadratic-like map whose straightening $\xi(f)$ is P_c. Since the conformal modulus of the annulus $\psi_c(D_\Delta) \setminus \overline{\psi_c(P_c^{-1}(D_\Delta))}$ depends only on $3K(\mu)$, we obtain that there is a positive constant $\mu'(\mu)$ such that the conformal modulus of f_c is greater than or equal to $\mu'(\mu)$. If c is real then $f_c(\overline{z}) = \overline{f_c(z)}$, which implies that f_c is a real quadratic-like map.

For the parameter $c(f)$, the map $\psi_{c(f)} \circ \Phi_f$ is 1 quasiconformal and fixes three points (0, 1 and ∞). Therefore, $\psi_{c(f)} \circ \Phi_f$ is the identity map, and since the map $\psi_{c(f)} \circ \Phi_f$ conjugates f with $f_{c(f)}$, we get $f_{c(f)} = f$.

Now, let us prove that the quadratic-like map f_c satisfies inequality (7). By compactness of the set of all $3K(\mu)$ quasiconformal homeomorphisms ϕ on $\overline{\mathbb{C}}$ fixing three points (0, 1 and ∞), there are positive constants $l(s,\mu) \le s \le L(s,\mu)$ for every $s > 0$ with the property that

$$D_l \subset \phi(D_s) \text{ and } \overline{\mathbb{C}} \setminus \overline{D_L} \subset \phi(\overline{\mathbb{C}} \setminus \overline{D_s}) \ . \tag{8}$$

Thus, there is $\Delta'' = L(L(\Delta))$ with the property that $\omega_c = \psi_c \circ \psi_{c(f)}^{-1}$ is holomorphic in $\overline{\mathbb{C}} \setminus \overline{D_{\Delta''}}$ for every $c \in D_r(c(f))$, and $c(f) \in [1,2]$.

Let $S_{2\Delta''}$ be the circle centered at the origin and with radius $2\Delta''$. By (8), we obtain that $\omega_c(S_{2\Delta''})$ is at a uniform distance from 0 and ∞ for every $c \in D_r(c(f))$, and $c(f) \in [1,2]$. Hence, by the Cauchy Integral Formula, and since ω_c is a holomorphic motion over $D_r(c(f))$, the value $a_c = \omega_c'(\infty)$ varies holomorphically with c, and there is a constant $L_1(\mu) > 0$ with the property that

$$|a_c - 1| < L_1 |c - c(f)| \ . \tag{9}$$

Thus, (i) the map $a_c\omega_c$ is holomorphic in $\overline{\mathbb{C}} \setminus \overline{D_{\Delta''}}$; (ii) $||\mu_{a_c\omega_c}||_\infty$ is less than or equal to $(9K^2-1)/(9K^2+1)$; and (iii) $\lim_{|z|\to\infty}(a_c\omega_c(z) - z) = 0$. Hence, by Theorem 9, there is a positive constant $L_2(\mu)$ such that, for every $c \in D_r(c(f))$, and for every $c(f) \in [1,2]$, we get

$$||a_c\omega_c - id||_{C^0} \le L_2 ||\mu_{a_c\omega_c}||_\infty \ . \tag{10}$$

Since $a_c\omega_c$ is a holomorphic motion over $D_r(c(f))$, and by Theorem 10, we get

$$||\mu_{a_c\omega_c}||_\infty \le \frac{|c - c(f)|}{r} \ . \tag{11}$$

By inequalities (9), (10), and (11) there is a positive constant $L_3(\mu)$ such that, for every $c(f) \in [1,2]$, and for every $c \in (c(f) - r, c(f) + r)$, we obtain

$$||\omega_c - id||_{C^0(I)} < L_3 |c - c(f)| \ . \tag{12}$$

This implies that

$$||\omega_c^{-1} - id||_{C^0(I)} < L_3 |c - c(f)| \ . \tag{13}$$

Since ω_c is a $9K^2$ quasiconformal homeomorphism, and fixes three points, we obtain from Theorem 4.3 in page 70 of [10] that there are positive constants $\beta(\mu) \leq 1$ and $L_4(\mu)$ with the property that $||\omega_c||_{C^\beta(I)} < L_4$. Then by inequalities (12) and (13) there is a positive constant $L_5(\mu)$ such that, for every $c(f) \in [1,2]$, and for every $c \in (c(f)-r, c(f)+r)$, we have

$$\begin{aligned} ||f_c - f_{c(f)}||_{C^0(I)} &\leq ||\omega_c - id||_{C^0(I)} + ||\omega_c||_{C^\beta(I)}||P_c - P_{c(f)}||^\beta_{C^0(I)} \\ &\quad + ||\omega_c||_{C^\beta}||P_{c(f)}||^\beta_{C^1(I)}||\omega_c^{-1} - id||^\beta_{C^0(I)} \\ &\leq L_5|c - c(f)|^\beta \ . \end{aligned}$$

Finally, by increasing the constant L_5 if necessary, we obtain that the last inequality is also satisfied for every $c(f)$ and c contained in $[1,2]$. □

4. Proofs of the main results

4.1. **Proof of Lemma 2.** Let $f = \phi_f \circ p$ be a C^2 infinitely renormalizable map with bounded combinatorial type. Let N be such that the combinatorial type of f is bounded by N, and set $b = ||\phi_f||_{C^2}$. By Lemma 8, there are positive constants $\gamma(N) < 1$, $\alpha(N)$, $\mu(N)$, and $c_1(b,N)$ with the following properties: for every $n \geq 0$, there is an $[\alpha n + 1]$ times renormalizable quadratic-like map F_n with renormalization type $\underline{\sigma}(n) = \sigma_{R^n f}, \ldots, \sigma_{R^{n+[\alpha n]} f}$, with conformal modulus greater than or equal to μ, and satisfying

$$||R^n f - F_n||_{C^0(I)} \leq c_1 \gamma^n \ . \tag{14}$$

By Milnor-Thurston's topological classification (see [14] and Theorem 4.2a. in page 470 of [18]), the real values c for which the real quadratic maps $P_c(z) = 1 - cz^2$ have renormalization type $\underline{\sigma}(n)$ is an interval $I_{\underline{\sigma}(n)}$. Thus, by Sullivan's pull-back argument (see [21] and Theorem 4.2b. in page 471 of [18]), there is a unique $c_n \in I_{\underline{\sigma}(n)}$ such that P_{c_n} has the same combinatorial type as $R^n(f)$. By Douady-Hubbard's Theorem 1 in [6], there is a unique quadratic map $\xi(F_n) = P_{c(F_n)}$ which is hybrid conjugated to F_n. Since F_n has renormalization type $\underline{\sigma}(n)$, the parameter $c(F_n)$ belongs to $I_{\underline{\sigma}(n)}$. By Lyubich's Theorem 9.6 in page 79 of [13], there are positive constants $\lambda(N) < 1$ and $c_2(N)$ such that $|I_{\underline{\sigma}(n)}| \leq c_2 \lambda^n$. Therefore, $|c_n - c(F_n)| \leq c_2 \lambda^n$.

By Lemma 12, there are positive constants $\beta(\mu) < 1$, $D(\mu)$, and $\mu'(\mu)$ with the following properties: for every $n \geq 0$, there is a real quadratic-like map f_n with conformal modulus greater than or equal to μ', satisfying $\xi(f_n) = P_{c_n}$, and such that

$$||f_n - F_n||_{C^0(I)} \leq D|c_n - c(F_n)|^\beta \leq D c_2^\beta \lambda^{\beta n} \ .$$

Therefore, the map f_n has the same combinatorial type as $R^n(f)$, and, by inequality (14), for $C(b,N) = c_1 + Dc_2^\beta$ and $\eta(N) = \max\{\gamma, \lambda^\beta\}$, we get

$$||R^n f - f_n||_{C^0(I)} \leq C\eta^n \ .$$

□

4.2. **Proof of Theorem 1.** Let $f = \phi_f \circ p$ and $g = \phi_g \circ p$ be any two C^2 infinitely renormalizable unimodal maps with the same bounded combinatorial type. Let N be such that the combinatorial type of f and g are bounded by N, and set $b = \max\{||\phi_f||_{C^2}, ||\phi_g||_{C^2}\}$. For every $n \geq 0$, let $m = [\alpha n]$, where $0 < \alpha < 1$ will be fixed later in the proof. By Lemma

2, there are positive constants $\eta(N) < 1$ and $c_1(b, N)$, and there are infinitely renormalizable real quadratic-like maps F_m and G_m with the following property:

$$||R^m f - F_m||_{C^0(I)} \leq c_1 \eta^{\alpha n} \text{ and } ||R^m g - G_m||_{C^0(I)} \leq c_1 \eta^{\alpha n} \ . \tag{15}$$

By Lemma 6, there are positive constants $n_3(b)$ and $L(N)$ such that, for every $m > n_3$, we get

$$\begin{aligned} ||R^n f - R^{n-m} F_m||_{C^0(I)} &\leq L^{n-m} ||R^m f - F_m||_{C^0(I)} \\ &\leq c_1 \left(L^{1-\alpha} \eta^{\alpha}\right)^n \ , \end{aligned} \tag{16}$$

and, similarly,

$$||R^n g - R^{n-m} G_m||_{C^0(I)} \leq c_1 \left(L^{1-\alpha} \eta^{\alpha}\right)^n \ . \tag{17}$$

Now, we fix $0 < \alpha(N) < 1$ such that $L^{1-\alpha}\eta^{\alpha} < 1$.

Again, by Lemma 2, F_m and G_m have conformal modulus greater than or equal to $\mu(N)$, and the same combinatorial type as $R^m f$ and $R^m g$. Therefore, by McMullen's Theorem 9.22 in page 172 of [16], there are positive constants $\nu_2(N) < 1$ and $c_2(\mu, N)$ with the property that

$$||R^{n-m} F_m - R^{n-m} G_m||_{C^0(I)} \leq c_2 \nu_2^{n-m} \ . \tag{18}$$

By inequalities (16), (17), and (18), there are constants $c_3(b, N) = 2c_1 + c_2$ and $\nu_3(N) = \max\{L^{1-\alpha}\eta^{\alpha}, \nu_2^{1-\alpha}\}$ such that

$$||R^n f - R^n g||_{C^0(I)} \leq c_3 \nu_3^n \ .$$

By Theorem 9.4 in page 552 of [18], the exponential convergence implies that there is a $C^{1+\alpha}$ diffeomorphism which conjugates f and g along the closure of the corresponding orbits of the critical points for some $\alpha(N) > 0$. □

The exponential convergence of the renormalization operator in the space of real analytic unimodal maps holds for every combinatorial type. Indeed, if f and g are real analytic infinitely renormalizable maps, by the complex bounds in Theorem A of Levin-van Strien in [11], there exists an integer N such that $R^N(f)$ and $R^N(g)$ have quadratic like extensions. Then we can use Lyubich's Theorem 1.1 in [12] to conclude the exponential convergence. However, as we pointed out before, this is not sufficient to give the $C^{1+\alpha}$ rigidity. Finally, at the moment, we cannot prove the exponential convergence of the operator for C^2 mappings with unbounded combinatorics.

Acknowledgments Alberto Adrego Pinto would like to thank IMPA, University of Warwick, and IMS at SUNY Stony Brook for their hospitality. We would like to thank Edson de Faria, and Mikhail Lyubich for useful discussions. This work has been partially supported by the Pronex Project on Dynamical Systems, Fundação para a Ciência, Praxis XXI, Calouste Gulbenkian Foundation, and Centro de Matemática Aplicada, da Universidade do Porto, Portugal.

References

1. L. V. Ahlfors, *Lectures on quasiconformal mappings.* D. van Nostrand Company, Inc. Princeton 1966.
2. L. V. Ahlfors, *Conformal invariants, topics in geometric function theory.* Mc Graw-Hill, New York 1973.
3. L. V. Ahlfors and L. Bers, Riemann's mapping theorem for variable metrics. *Annals of Math.* (2) **72** (1960), 385-404.
4. P. Coullet and C. Tresser, Itération d'endomorphismes et groupe de renormalisation. *J. Phys.* Colloque C **539**, C5-25 (1978).
5. A Douady, Prolongement des mouvements holomorphes [d'aprés Słodkowski et autres]. In *Séminaire Bourbaki (Nov. 93)*, 7-20. Astérisque, v. 227-228, 1995.
6. A. Douady and J. H. Hubbard, On the dynamics of polynomial-like maps. *Ann. Sci. Éc. Norm. Sup.*, **18** (1985), 287-343.
7. E. de Faria and W. de Melo, Rigidity of critical circle maps I. *IMS Stony Brook preprint* 1997/16 (1997).
8. M. J. Feigenbaum, Quantitative universality for a class of nonlinear transformations. *J. Stat. Phys.* **19** (1978), 25-52.
9. O. Lehto, *Univalent functions and Teichmüller spaces.* Graduate Texts in Mathematics 109. Springer-Verlag, 1987.
10. O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane.* Springer-Verlag (1973).
11. G. Levin and S. van Strien, Local connectivity of the Julia set of real polynomials. *Annals of Math.* **147** (1998), 471-541.
12. M. Lyubich, Almost Every real quadratic map is either regular or stochastic. *IMS Stony Brook preprint* 1997/8 (1997).
13. M. Lyubich, Feigenbaum-Coullet-Tresser Universality and Milnor's Hairiness conjecture. *IHES Preprint*, 1-31. To be published in *Annals of Math.*
14. J. Milnor, The monotonicity theorem for real quadratic maps. *Mathematische Arbeitstagung Bonn* 1983.
15. C. McMullen, *Complex dynamics and renormalization.* Annals of Math. Studies, v. 135. Princeton University press, 1994.
16. C. McMullen, *Renormalization and 3-Manifolds which Fiber over the Circle.* Annals of Math. Studies, v. 142. Princeton University press, 1996.
17. W. de Melo Rigidity and renormalization in the one dimensional dynamical systems. In *Proceedings of the International Congress of Mathematicians, Berlin 1998*, 765-779. Documenta Mathematica 1998.
18. W. de Melo and S. van Strien *One-Dimensional Dynamics.* A Series of Modern Surveys in Mathematics, Springer-Verlag, 1993.
19. A. A. Pinto and D. Rand, Global phase space universality, smooth conjugacies and renormalization:2. The $C^{k+\alpha}$ case using rapid convergence of Markov families. *Nonlinearity* **4** (1991) 1-31.
20. D. Rand, Global phase space universality, smooth conjugacies and renormalization: 1. The $C^{1+\alpha}$ case. *Nonlinearity* **1** (1988) 181-202.
21. D. Sullivan, Bounds, quadratic differentials, and renormalization conjectures. AMS Centennial Publications. In *Volume 2: Mathematics into the Twenty-first Century (1988 Centennial Symposium, August 8-12).* American Mathematical Society, Providence, RI.,1991.
22. D. Sullivan, Linking the universalities of Milnor-Thurston, Feigenbaum and Ahlfors-Bers. In L. R. Goldberg and A. V. Phillips, editors, *Topological Methods in Modern Mathematics*, 543-563, Publish or Perish, Inc.1993.
23. Z. Słodkowski, Holomorphic motions and polynomial hulls. *Proc. Amer. Math. Soc.*, **111** (1991), 347-355.

(W. de Melo) IMPA, Rio de Janeiro, Brazil.
E-mail address, W. de Melo: `demelo@impa.br`

(A. A. Pinto) DMA, Faculdade de Ciências, Universidade do Porto, 4000 Porto, Portugal.
E-mail address, A. A. Pinto: `aapinto@fc.up.pt`